ON THE SATURATION SEQUENCE OF THE RATIONAL NORMAL CURVE

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ABSTRACT: Let $C \subseteq \mathbf{P}^d$ denote the rational normal curve of order d. Its homogeneous defining ideal $I_C \subseteq \mathbf{Q}[a_0, \dots, a_d]$ admits an SL_2-stable filtration $J_2 \subseteq J_4 \subseteq \dots \subseteq I_C$ by sub-ideals such that the saturation of each J_{2q} equals I_C. Hence, one can associate to d a sequence of integers $(\alpha_1, \alpha_2, \dots)$ which encodes the degrees in which the successive inclusions in this filtration become trivial. In this paper we establish several lower and upper bounds on the α_q, using *inter alia* the methods of classical invariant theory.

Keywords: covariants, Gordan's syzygies, rational normal curve, saturation.

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1. INTRODUCTION

1.1. The rational normal curve of order d in $\mathbf{P}^d$ and its homogeneous defining ideal usually make an obligatory appearance in textbooks on algebraic geometry[1]. This is not without its reasons. The latter admits a winsome description as the ideal of maximal minors of a $2 \times d$ matrix of variables, usually called the catalecticant matrix (see §1.7 below).

However, this formulation disguises the fact that the ideal carries a nontrivial filtration which is invariant under the automorphisms of $\mathbf{P}^d$ fixing the curve. The object of this paper is to initiate a study of this filtration; the main results are described in §1.9 after the required notation is available.

Throughout, the base field will be $\mathbf{Q}$ (the field of rational numbers). Classical treatments of the necessary background in invariant theory may be found in [7, 14], and more modern treatments in [5, 11, 12, 13, 16].

[1] For instances, see [6, Exer. A2.10], [8, Lecture 1], [9, Ch. IV, Exer. 3.4].

1.2. **Transvectants.** Let $A(x_1, x_2)$ and $B(x_2, x_2)$ denote binary forms of orders p, q respectively in the variables $\mathbf{x} = \{x_1, x_2\}$. Their r-th transvectant[2] is defined by the formula

$$(A, B)_r = \frac{(p-r)!\,(q-r)!}{p!\,q!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r A}{\partial x_1^{r-i}\,\partial x_2^i} \frac{\partial^r B}{\partial x_1^i\,\partial x_2^{r-i}}; \qquad (1)$$

for $0 \leqslant r \leqslant \min(p, q)$. It is of order $p + q - 2r$ in $\mathbf{x}$. If $r > \min(p, q)$, then $(A, B)_r = 0$. Moreover, $(A, B)_r = (-1)^r (B, A)_r$, and hence $(A, A)_r$ vanishes for odd values of r.

1.3. **Representations of SL_2.** For $p \geqslant 0$, let S_p denote the set of binary forms of order p in $\mathbf{x}$ (with coefficients in $\mathbf{Q}$). The group $SL_2\,\mathbf{Q}$ acts on S_p as follows: for $g = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in SL_2$,

$$A(x_1, x_2) \xrightarrow{g} A(\gamma_{11}\,x_1 + \gamma_{12}\,x_2, \gamma_{21}\,x_1 + \gamma_{22}\,x_2).$$

Up to isomorphism, $\{S_p : p \geqslant 0\}$ is the set of all the finite-dimensional irreducible representations of SL_2, and each such representation splits as a direct sum of irreducibles (see [13, Ch. 10]). For any $p, q \geqslant 0$, there is a decomposition

$$S_p \otimes S_q \simeq \bigoplus_{r=0}^{\min(p,q)} S_{p+q-2r},$$

and the image of $A \otimes B$ via the projection map $S_p \otimes S_q \longrightarrow S_{p+q-2r}$, is the transvectant $(A, B)_r$. There is an isomorphism of S_p with its dual representation $S_p^* = \mathrm{Hom}(S_p, \mathbf{Q})$, which associates $A \in S_p$ with the functional $B \longrightarrow (A, B)_p$.

1.4. **The ring of covariants.** Fix an integer $d \geqslant 1$, and introduce variables $a_0, \dots, a_d$. Define the bigraded polynomial ring

$$\mathcal{C} = \mathbf{Q}[a_0, \dots, a_d; x_1, x_2] = \bigoplus_{m,n \geqslant 0} \mathcal{C}_{m,n},$$

where m (respectively n) denotes the degree in the a-variables (respectively $\mathbf{x}$-variables). Let

$$\mathbb{F} = \sum_{i=0}^{d} \binom{d}{i} a_i\, x_1^{d-i}\, x_2^i \in \mathcal{C}, \qquad (2)$$

denote the generic binary d-ic, and define $\mathcal{A}$ to be the smallest $\mathbf{Q}$-subalgebra of $\mathcal{C}$ satisfying the following two properties:

[2] Usually r is called the index of transvection.

- $\circ$ $\mathbb{F} \in \mathcal{A}$,
- $\circ$ if $T, T' \in \mathcal{A}$ are bihomogeneous elements, then $(T, T')_r \in \mathcal{A}$ for all $r \geqslant 0$.

In other words, $\mathcal{A}$ is spanned as a $\mathbf{Q}$-vector space by all compound transvectant expressions

$$(\mathbb{F}, \mathbb{F})_2,\ (\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_5,\ ((\mathbb{F}, \mathbb{F})_2, (\mathbb{F}, \mathbb{F})_4)_3, \dots \text{etc.}$$

We have a bigraded decomposition,

$$\mathcal{A} = \bigoplus_{m,n} \mathcal{A}_{m,n}, \quad \text{where } \mathcal{A}_{m,n} = \mathcal{C}_{m,n} \cap \mathcal{A}.$$

In classical literature $\mathcal{A}$ is called the ring of covariants[3] (of a binary d-ic); and an element $\Phi \in \mathcal{A}_{m,n}$ is called a covariant of degree m and order n. E.g., $(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_5$ is a covariant of degree 3 and order $3d - 14$. A covariant of order zero is called an invariant.

It is a fundamental result due to Gordan that $\mathcal{A}$ is finitely generated as a $\mathbf{Q}$-algebra (see [7, Ch. VI]). E.g., if $d = 4$, then $\mathcal{A}$ is generated by the elements

$$\mathbb{F},\ (\mathbb{F}, \mathbb{F})_2,\ (\mathbb{F}, \mathbb{F})_4,\ (\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_1,\ (\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_4;$$

of degree-orders $(1,4), (2,4), (2,0), (3,6), (3,0)$ respectively.

1.5. Now identify the generic form $\mathbb{F}$ with the natural trace element in $S_d^* \otimes S_d \simeq S_d \otimes S_d$; this amounts to letting $a_i = \frac{1}{d!}\,(-x_1)^i x_2^{d-i} \in S_d$. Then $R = \mathbf{Q}[a_0, \dots, a_d]$ is identified with the symmetric algebra $\bigoplus\limits_{m \geqslant 0} \operatorname{Sym}^m S_d$. Consider the decomposition

$$R_m \simeq \operatorname{Sym}^m S_d \simeq \bigoplus_n \, (S_n \otimes \mathbf{Q}^{\eta_{m,n}}).$$

A covariant $\Phi = \varphi_0\, x_1^n + \varphi_1\, x_1^{d-1}\, x_2 + \dots + \varphi_n\, x_2^n$ of degree-order (m,n) gives an SL_2-equivariant morphism

$$S_n \longrightarrow R_m, \quad A \longrightarrow (A, \Phi)_n;$$

and conversely, every such morphism arises from a covariant. Hence

$$\dim \mathcal{A}_{m,n} = \eta_{m,n} = \dim \operatorname{Hom}_{SL_2}(S_n, R_m).$$

[3]It is more common to define it as the invariant subring $\mathcal{C}^{SL_2}$, but our definition is equivalent.

E.g., for $d = 6$, there is a decomposition

$$R_3 \simeq \text{Sym}^3 S_6 \simeq S_{18} \oplus S_{14} \oplus S_{12} \oplus S_{10} \oplus S_8 \oplus (S_6 \otimes \mathbf{Q}^2) \oplus S_2;$$

in particular, $\dim \mathcal{A}_{3,6} = 2$. It is easy to verify that

$$\{(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_4, (\mathbb{F}, (\mathbb{F}, \mathbb{F})_4)_2\}$$

is a basis of $\mathcal{A}_{3,6}$. By contrast, since $\mathcal{A}_{3,8}$ is one-dimensional, the forms $(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_3$ and $(\mathbb{F}, (\mathbb{F}, \mathbb{F})_4)_1$ must be dependent; in fact there is an identical relation $7\,(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_3 - (\mathbb{F}, (\mathbb{F}, \mathbb{F})_4)_1 = 0$. Such calculations in $\mathcal{A}$ can be carried out by using the classical symbolic calculus (see [7]).

1.6. **Quadratic covariants.** Now let $e_d = [\frac{d}{2}]$, and write

$$\mathbb{H}_{2q} = (\mathbb{F}, \mathbb{F})_{2q}, \qquad \text{for } 1 \leqslant q \leqslant e_d,$$

which is a covariant of degree 2 and order $2d - 4q$. (Usually $\mathbb{H}_2$ is called the Hessian of $\mathbb{F}$.) We have a decomposition

$$R_2 \simeq \text{Sym}^2 S_d \simeq \bigoplus_{q=0}^{e_d} S_{2d-4q},$$

in which the summand S_{2d-4q} corresponds to the span of the coefficients of $\mathbb{H}_{2q}$. Define W_{2q} to be the subspace of R_2 generated by all the coefficients of $\mathbb{H}_2, \mathbb{H}_4, \dots, \mathbb{H}_{2q}$, and let J_{2q} be the ideal in R generated by W_{2q}. This defines a filtration

$$J_2 \subsetneq J_4 \subsetneq \cdots \subsetneq J_{2e_d}, \tag{3}$$

which is nontrivial for all $d \geqslant 4$.

1.7. Now $\mathbf{P}S_d = \text{Proj}\, R$ is the space of binary d-ics (distinguished up to scalars). It is a classical result (see [12, Proposition 2.23]) that the following conditions are equivalent for $A \in S_d$.

(1) $(A, A)_2 = 0$.
(2) $(A, A)_2 = (A, A)_4 = \cdots = (A, A)_{2e_d} = 0$.
(3) There exists a linear form $t_1\, x_1 + t_2\, x_2$, such that $A = (t_1\, x_1 + t_2\, x_2)^d$.

It follows that the variety cut out by the ideal J_{2e_d} is the rational normal curve $C = \{[(t_1\, x_1 + t_2\, x_2)^d\,] : t_1, t_2 \in \mathbf{Q}\} \subseteq \mathbf{P}S_d$. Since the defining ideal $I_C \subseteq R$ is SL_2-stable and generated by quadrics, in fact $J_{2e_d} = I_C$. It may also be described as the ideal of maximal minors of the catalecticant matrix

$$\begin{bmatrix} a_0 & a_1 & \dots & a_{d-2} & a_{d-1} \\ a_1 & a_2 & \dots & a_{d-1} & a_d \end{bmatrix}.$$

The equivalence (1) $\iff$ (3) implies that J_2 defines C set-theoretically, but in fact a stronger statement holds.

Proposition 1.1. *The saturation of J_2 equals I_C.*

PROOF. See [1, Lemma 3.1], as well as §2.4 below. $\square$

1.8. It follows that all the ideals J_{2q} coincide in sufficiently high degrees. For $1 \leqslant q \leqslant e_d - 1$, define

$$\alpha_q = \min \{m : (J_{2q})_t = (J_{2q+2})_t \text{ for all } t \geqslant m\},$$

then $(\alpha_1, \dots, \alpha_{e_d-1})$ will be called the **saturation sequence** of d. I am enclosing the table of saturation sequences for $d \leqslant 20$. It was calculated in MACAULAY-2.

d	saturation sequence
4	(3)
5	(3)
6	$(5, 3)$
7	$(4, 3)$
8	$(5, 3, 3)$
9	$(5, 3, 3)$
10	$(5, 3, 3, 3)$
11	$(5, 3, 3, 3)$
12	$(7, 5, 3, 3, 3)$
13	$(5, 4, 3, 3, 3)$
14	$(7, 5, 3, 3, 3, 3)$
15	$(6, 5, 3, 3, 3, 3)$
16	$(7, 5, 4, 3, 3, 3, 3)$
17	$(7, 5, 4, 3, 3, 3, 3)$
18	$(7, 5, 5, 3, 3, 3, 3, 3)$
19	$(7, 5, 4, 3, 3, 3, 3, 3)$
20	$(8, 5, 5, 4, 3, 3, 3, 3, 3)$

Recall that the *satiety* of J_{2q} is defined to be the integer (cf. [3, p. 593])

$$\min\{\, m : (J_{2q})_t = (I_C)_t \text{ for all } t \geqslant m\}.$$

It is equal to $\max \{\alpha_q, \alpha_{q+1}, \dots, \alpha_{e_d-1}\}$.

1.9. **A summary of results.** Define

$$\mathfrak{S}(d) = \max\{\alpha_1, \alpha_2, \dots, \alpha_{e_d - 1}\},$$

which is the satiety of J_2, and let

$$\zeta(d) = \frac{1}{d-2}\sqrt{\frac{(d-1)(d^2-2)}{2}}.$$

Theorem 1.2. *For $d \geqslant 4$, we have inequalities*

$$\zeta(d) \leqslant \mathfrak{S}(d) \leqslant d+2.$$

Broadly speaking, the lower bound implies that $\mathfrak{S}(d)$ grows no slower than $\sqrt{\frac{d}{2}}$. It will be proved in §2.1. A proof of the upper bound is given in §2.2.

The next theorem (which is merely an aggregate of separate propositions) establishes some specific lower bounds for α_1, α_2 and α_3.

Theorem 1.3. *Let (q, b, N) denote any of the following triples:*

$$(1,3,6), \quad (1,4,8), \quad (2,3,12), \quad (3,3,16).$$

Then $\alpha_q > b$ for all $d \geqslant N$.

The proofs are given in §3.3.

The following theorem was inspired by the observation that the saturation sequences tend to end in long strings of 3s. Let

$$\begin{aligned} &N_1 = 4, \quad N_2 = 8, \quad N_3 = 10, \quad N_4 = 14, \\ &N_5 = 18, \quad N_6 = 22, \quad N_7 = 26, \quad N_8 = 30. \end{aligned} \tag{4}$$

Theorem 1.4. *Let s and d be integers such that $1 \leqslant s \leqslant 8$, and $d \geqslant N_s$. Then at least the last s integers in the saturation sequence of d are all equal to 3.*

The proof is based upon Gordan's cubic syzygies. It will be given in §4.

In the proofs of the results above, I have had to use machine calculations in order to find some complicated compound transvectants, and to evaluate some large determinants. They were all done in MAPLE.

The following two conjectures arise naturally from the previous table. I have been unable to make any progress on either of them.

Conjecture 1.5. *The saturation sequence is non-increasing. (This would imply that $\mathfrak{S}(d) = \alpha_1$.)*

Conjecture 1.6. *For all $d \geqslant 6$, there is always a strict inequality $\alpha_1 > \alpha_2$.*

2. BOUNDS ON $\mathfrak{S}(d)$

2.1. In this section we will prove the lower bound on $\mathfrak{S}(d)$. Assume that $(J_2)_m = (I_C)_m$ for some $m > 2$. Then the natural morphism

$$W_2 \otimes R_{m-2} \longrightarrow (I_C)_m$$

must be surjective, hence by counting dimensions we must have

$$(2d-3)\binom{m+d-2}{d} \geqslant \binom{m+d}{d} - (m\,d+1). \tag{5}$$

One should like to force a lower bound on m from this inequality. This is carried out in the following proposition, which I owe to my colleague A. Abdesselam. Although the proof is elementary in essence, some tricky manipulations are involved.

Proposition 2.1. *If $m < \zeta(d)$, then the inequality in (5) is false.*

PROOF. Transfer the right-hand side of (5) to the left-hand side, and multiply by $d!$. Thus (5) is equivalent to

$$(2d-3)\left(\prod_{k=m-1}^{m+d-2} k\right) - \left(\prod_{k=m+1}^{m+d} k\right) + d!\,(m\,d+1) \geqslant 0,$$

or what is the same,

$$\underbrace{(2d-3)(m-1)m - (m+d-1)(m+d)}_{Q(d,m)} \times \left(\prod_{k=m+1}^{m+d-2} k\right) + d!\,(m\,d+1) \geqslant 0. \tag{6}$$

We have a factorisation

$$Q(d,m) = 2(d-2)(m - \xi_1(d))(m - \xi_2(d)),$$

where

$$\xi_1(d) = \frac{d-1}{d-2} - \zeta(d), \quad \xi_2(d) = \frac{d-1}{d-2} + \zeta(d).$$

It is easy to see that $\xi_1(d) < 0$ and $\xi_2(d) > 0$.

Case $m = 3$. After substitution, the left-hand side of (6) becomes

$$\begin{aligned} &-(d^2 - 7\,d + 24)\,\frac{d!\,(d+1)}{6} + d!\,(3\,d+1) \\ &= -\frac{d!}{6}\,(d-2)\,(d-2-\sqrt{13})\,(d-2+\sqrt{13}). \end{aligned} \tag{7}$$

Now assume $3 < \zeta(d)$. Then

$$3(d-2) < \sqrt{\frac{(d-1)(d^2-2)}{2}} < \sqrt{\frac{(d-1)(d^2-1)}{2}} = (d-1)\sqrt{\frac{d+1}{2}},$$

and since $\frac{d-1}{d-2} \leqslant \frac{3}{2}$ for $d \geqslant 4$, we have

$$3 < \frac{3}{2}\sqrt{\frac{d+1}{2}}.$$

This implies that $d > 7$, hence (7) is negative.

Case $m \geqslant 4$**.** Assume $m < \zeta(d)$; then $\xi_1(d) < 0 < m < \xi_2(d)$, which implies that $Q(d,m) < 0$. We want to show that left-hand side of (6) is negative. Replace $m\,d+1$ by the larger quantity $(m+1)\,d$ and divide by $m+1$ to get

$$Q(d,m) \times \underbrace{\left(\prod_{k=m+2}^{m+d-2} k\right)}_{T_m} + d! \times d. \tag{8}$$

It would be sufficient to show that (8) is negative. Observe that

$$\frac{T_{m+1}}{T_m} = \frac{m+d-1}{m+2} > 1,$$

i.e., T_m increases with m. Hence, (8) is bounded above by the quantity

$$Q(d,m)\,T_4 + d! \times d = Q(d,m)\,\frac{(d+2)!}{120} + d! \times d. \tag{9}$$

Since $m - \xi_1(d) > 4$, and $m - \xi_2(d) < \zeta(d) - \xi_2(d) < -1$, we get $Q(d,m) < -8(d-2)$. Thus (9) is strictly smaller than

$$-8\,(d-2)\,\frac{(d+2)!}{120} + d! \times d = -\frac{1}{15}(d-4)(d^2+5d+1)\,d! < 0.$$

The proposition is proved. □

2.2. **The Koszul complex.** The upper bound on $\mathfrak{S}(d)$ will be established by a spectral sequence argument. (Compare the proof of Theorem 1 in [15].) We refer to [9, Ch. III.5] for standard results on the cohomology of line bundles on $\mathbf{P}^d$.

The subspace $W_2 \subseteq R_2$ gives a morphism

$$S_{2d-4} \otimes \mathcal{O}_{\mathbf{P}^d}(-2) \xrightarrow{\partial} \mathcal{O}_{\mathbf{P}^d}.$$

By Proposition 1.1, we have, $\operatorname{im} \partial = \mathcal{I}_C$ (the ideal sheaf of C). Consider the Koszul complex of ∂, and replace $\mathcal{O}_{\mathbf{P}^d}$ with $\mathcal{I}_C$. This defines a complex $\mathcal{K}^\bullet$ of coherent $\mathcal{O}_{\mathbf{P}^d}$-modules

$$0 \to \mathcal{K}^{-(2d-3)} \to \ldots \to \mathcal{K}^p \xrightarrow{h^p} \mathcal{K}^{p+1} \to \ldots \to \mathcal{K}^{-1} \xrightarrow{h^{-1}} \mathcal{K}^0 \to 0,$$

where

$$\mathcal{K}^p = \begin{cases} \wedge^{-p} S_{2d-4} \otimes \mathcal{O}_{\mathbf{P}^d}(2\,p) & \text{for } -(2d-3) \leqslant p \leqslant -1, \\ \mathcal{I}_C & \text{for } p = 0. \end{cases}$$

We will write $\mathcal{K}^\bullet(m)$ for $\mathcal{K}^\bullet \otimes \mathcal{O}_{\mathbf{P}^d}(m)$. Let $\mathcal{H}^p = \ker h^p / \operatorname{im} h^{p-1}$ denote the cohomology sheaves of $\mathcal{K}^\bullet$.

2.3. There are two second quadrant spectral sequences in the range

$$-(2d-3) \leqslant p \leqslant 0, \quad 0 \leqslant q \leqslant d,$$

which abut to the hypercohomology[4] of $\mathcal{K}^\bullet(m)$; namely

$$\begin{aligned} & E_2^{p,q} = H^q(\mathbf{P}^d, \mathcal{H}^p \otimes \mathcal{O}_{\mathbf{P}^d}(m)), \quad \delta_r : E_r^{p,q} \longrightarrow E_r^{p-r+1,q+r} \\ & E_\infty^{p,q} \Rightarrow \mathbb{H}^{p+q}(\mathcal{K}^\bullet(m)); \end{aligned} \tag{10}$$

and

$$\begin{aligned} & \widetilde{E}_1^{p,q} = H^q(\mathbf{P}^d, \mathcal{K}^p(m)), \quad \widetilde{\delta}_r : \widetilde{E}_r^{p,q} \longrightarrow \widetilde{E}_r^{p+r,q-r+1} \\ & \widetilde{E}_\infty^{p,q} \Rightarrow \mathbb{H}^{p+q}(\mathcal{K}^\bullet(m)). \end{aligned} \tag{11}$$

Henceforth, let

$$m = d + 2. \tag{12}$$

First, consider the terms in (10). The support of each $\mathcal{H}^p$ is contained in C (see [4, Prop. 1.6.5]), hence $E_2^{p,q} = 0$ for $q \geqslant 2$. This forces $E_2^{p,q} = E_\infty^{p,q}$. Since h^{-1} is a surjection, $\mathcal{H}^0 = 0$.

The sheaf $\mathcal{H}^{-1}$ will be calculated in Proposition 2.2 below, from which it will follow that $H^1(\mathbf{P}^d, \mathcal{H}^{-1} \otimes \mathcal{O}_{\mathbf{P}^d}(m)) = 0$. Hence $E_2^{p,q} = 0$ for all $p + q = 0$, implying that

$$\mathbb{H}^0(\mathcal{K}^\bullet(m)) = 0. \tag{13}$$

On the other hand, all the nonzero $\widetilde{E}_1^{p,q}$ terms in (11) are concentrated in the rows $q = 0, d$. Our choice of m ensures that $\mathcal{K}^{-(d+1)}, \mathcal{K}^{-d}$ are respectively equal to

$$\wedge^{d+1} S_{2d-4} \otimes \mathcal{O}_{\mathbf{P}^d}(-d), \quad \wedge^d S_{2d-4} \otimes \mathcal{O}_{\mathbf{P}^d}(-d+2),$$

[4] The hypercohomology groups are denoted by upper indices on $\mathbb{H}$. There is scarcely any danger of confusion with the covariants $\mathbb{H}_{2q}$, which do not appear in this section.

and hence $\widetilde{E}_1^{p,q} = 0$ for $(p,q) = (-d-1,d), (-d,d)$. On account of (13), this forces $\widetilde{E}_\infty^{0,0} = \widetilde{E}_2^{0,0} = 0$. Hence the morphism

$$H^0(\mathcal{K}^{-1}(m)) \longrightarrow H^0(\mathcal{K}^0(m))$$

must be surjective, i.e., $(J_2)_{d+2} = (I_C)_{d+2}$, and thus $\mathfrak{S}(d) \leqslant d+2$. $\square$

2.4. Consider the sheaf $\mathcal{H}^{-1} = \ker h^{-1}/\operatorname{im} h^{-2}$ supported on $C \simeq \mathbf{P}^1$. Henceforth we denote it by $\mathcal{H}$ for brevity. Since $\mathcal{K}^\bullet$ is an SL_2-equivariant complex, and the action of SL_2 on C is transitive, $\mathcal{H}$ must be torsion-free and hence locally free.

Proposition 2.2. *Assume $d \geqslant 3$. Then $\mathcal{H}$ is a rank $d-2$ vector bundle on $\mathbf{P}^1$. Moreover, it splits as a direct sum of line bundles $\oplus\, \mathcal{O}_{\mathbf{P}^1}(t)$, where each summand satisfies the inequalities $-4d+4 \leqslant t \leqslant -2d-2$.*

It follows that the group

$$H^1(\mathbf{P}^d, \mathcal{H} \otimes \mathcal{O}_{\mathbf{P}^d}(m)) \simeq \oplus\, H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(md+t)),$$

vanishes for $m \geqslant 4$, since $md + t \geqslant 4 > -2$. This suffices to conclude the argument in the previous section.

PROOF. The proof will follow from a calculation of local transition functions. Let $\lambda_i = a_i/a_0$ for $1 \leqslant i \leqslant d$, and $f = \mathbb{F}/a_0 = x_1^d + \sum\limits_i \binom{d}{i} \lambda_i\, x_1^{d-i}\, x_2^i$. We will write the Hessian $(f,f)_2$ as

$$\sum_{r=2}^{d} \binom{2d-4}{r-2} u_r\, x_1^{2d-r-2}\, x_2^{r-2} + \sum_{s=d+1}^{2d-2} \binom{2d-4}{s-2} v_s\, x_1^{2d-s-2}\, x_2^{s-2}; \quad (14)$$

where u_r, v_s are elements in the ring $\mathfrak{A} = \mathbf{Q}[\lambda_1, \dots, \lambda_d]$. (The rationale behind this notation will emerge below.) A direct calculation with formula (1) shows that we have expressions

$$\kappa\, u_2 = \lambda_2 - \lambda_1^2, \quad \kappa\, u_3 = \lambda_3 - \lambda_1\, \lambda_2,$$

and in general

$$\kappa\, u_r = \lambda_r - P_r(\lambda_1, \dots, \lambda_{r-1}),$$

for some polynomials P_r. (Throughout, we have used κ as a placeholder for various nonzero rational constants which need not be precisely specificed. See Example 2.4 below.) If we define the weight of λ_i to be i, then u_r, v_s are isobaric of weights r, s respectively.

A simple induction shows that $\kappa\, u_r \equiv \lambda_r - \lambda_1^r \mod (u_2, \dots, u_{r-1})$. It follows that $u_2, \dots, u_d$ is a regular sequence, and that $\mathfrak{a} = (u_2, \dots, u_d) \subseteq \mathfrak{A}$ is the defining ideal of the affine piece of C in spec $\mathfrak{A} \subseteq \mathbf{P}^d$.

Since $v_s \in \mathfrak{a}$, we must have identities of the form $v_s = \sum_{r=2}^{d} g_{s-r}\, u_r$, where $g_{s-r} \in \mathfrak{A}$ are isobaric of weight $s-r$. Fix one such an identity for each s, and let

$$z_s = V_s - \sum_{r=2}^{d} g_{s-r}\, U_r, \quad \text{for} \quad d+1 \leqslant s \leqslant 2d-2.$$

2.5. Let M denote the free $\mathfrak{A}$-module of rank $2d-3$ on basis elements

$$U_r = (-1)^r\, x_2^{2d-r-2}\, x_1^{r-2}, \quad V_s = (-1)^s\, x_2^{2d-s-2}\, x_1^{s-2},$$

for the same range of r, s as in (14). The notation is chosen in such a way that the complex $\mathcal{K}^{-2} \xrightarrow{h^{-2}} \mathcal{K}^{-1} \xrightarrow{h^{-1}} \mathcal{I}_C$ is represented over spec $\mathfrak{A}$ by the $\mathfrak{A}$-module maps

$$\wedge^2 M \xrightarrow{\tilde{\mathfrak{f}}} M \xrightarrow{\mathfrak{f}} \mathfrak{a},$$

where

$$\tilde{\mathfrak{f}}(W_i \wedge W_j) = w_j\, W_i - w_i\, W_j, \quad \text{and} \quad \mathfrak{f}(W_i) = ((f,f)_2, W_i)_{2d-4} = w_i.$$

(Here W stands for either U or V as dictated by the index i, and similarly for w. E.g., $W_2 = U_2, w_{d+1} = v_{d+1}$ etc.)

2.6. Since the $\mathfrak{A}$-module

$$N = \Gamma(\text{spec}\,\mathfrak{A}, \mathcal{H}) = \ker \mathfrak{f}/\text{im}\,\tilde{\mathfrak{f}}$$

is annihilated by $\mathfrak{a}$, it may be regarded as a module over $\mathfrak{A}/\mathfrak{a} \simeq \mathbf{Q}[\lambda]$. (We have written λ for λ_1.) It is clear that $z_s \in \ker \mathfrak{f}$. Let ξ_s denote the class of z_s in N.

Lemma 2.3. *With notation as above, N is the free $\mathbf{Q}[\lambda]$-module over the elements $\{\xi_s\}$.*

PROOF. If $z = \sum_r \alpha_r\, U_r + \sum_s \beta_s\, V_s \in \ker \mathfrak{f}$, then $z - \sum_s \beta_s\, z_s$ is an element in $\ker \mathfrak{f}$ which involves only the U_r. Hence it must necessarily lie in im $\tilde{\mathfrak{f}}$, since there are no syzygies between the u_r except those coming from the tautological Koszul relations. This shows that the $\{\xi_s\}$ generate N. Now consider the map

$$e : \mathbf{Q}[\lambda]^{d-2} \longrightarrow N, \quad p = (p_{d+1}(\lambda), \dots, p_{2d-2}(\lambda)) \longrightarrow \sum p_s(\lambda)\, \xi_s.$$

Assume $e(p) = 0$, and let s be the largest index such that $p_s(\lambda) \neq 0$. Then the weight s part of the relation gives an identity $p_s(0)\, \xi_s + \cdots = 0$. We may assume that $p_s(0) \neq 0$, since N is torsion-free. However, it is clear from the definition of $\widetilde{\mathfrak{f}}$ that no such element can lie in im $\widetilde{\mathfrak{f}}$. Hence $\ker e = 0$. $\square$

2.7. Now write $\mu_{-i} = a_{d-i}/a_d$ (considered to be of weight $-i$), and let $\mathfrak{A}' = \mathbf{Q}[\mu_{-1}, \dots, \mu_{-d}]$. If $f' = \mathbb{F}/a_d$, then $(f', f')_2 =$

$$\sum_{r=2}^{d} \binom{2d-4}{r-2} u_{-r}\, x_2^{2d-r-2}\, x_1^{r-2} + \sum_{s=d+1}^{2d-2} \binom{2d-4}{s-2} v_{-s}\, x_2^{2d-s-2}\, x_1^{s-2};$$

where $u_{-r}, v_{-s} \in \mathfrak{A}'$ are isobaric elements of weights $-r, -s$ respectively. The same results are true *mutatis mutandis* over spec $\mathfrak{A}'$, and we have generators $\{\xi_{-s}\}$ of N' with weights $-(2d-2), \dots, -(d+1)$. Define the vectors

$$\xi^+ = \begin{bmatrix} \xi_{d+1} \\ \vdots \\ \xi_{2d-2} \end{bmatrix}, \quad \xi^- = \begin{bmatrix} \xi_{-(2d-2)} \\ \vdots \\ \xi_{-(d+1)} \end{bmatrix}.$$

Then $\lambda^{-(3d-1)}\, \xi^+$ and ξ^- are two bases of $\Gamma(\text{spec}\,\mathfrak{A} \cap \text{spec}\,\mathfrak{A}', \mathcal{H})$ as a $\mathbf{Q}[\lambda, \lambda^{-1}]$-module, and hence there is a matrix $Q \in GL(d-2, \mathbf{Q}[\lambda, \lambda^{-1}])$ such that $Q\, \xi^- = \lambda^{-(3d-1)}\, \xi^+$. By taking the weights into account, one sees that the (i, j)-th entry of Q is of the form $c\, \lambda^{i-j}$ for some $c \in \mathbf{Q}$.

Now apply [10, Proposition 3.1] to Q. It produces a factorisation $Q = E^{-1}\, D\, F$, where

$$E \in GL(d-2, \mathbf{Q}[\lambda]), \quad F \in GL(d-2, \mathbf{Q}[\lambda^{-1}]),$$

and D is a diagonal matrix of the form $\begin{bmatrix} \lambda^{k_1} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda^{k_{d-2}} \end{bmatrix}$. Since the entries of $\lambda^{d-3}\, Q$ and $\lambda^{-(d-3)}\, Q$ are respectively in $\mathbf{Q}[\lambda]$ and $\mathbf{Q}[\lambda^{-1}]$, we have $-(d-3) \leqslant k_i \leqslant d-3$. Hence we have an identity

$$F\, \xi^- = \begin{bmatrix} \lambda^{t_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda^{t_{d-2}} \end{bmatrix} E\, \xi^+,$$

where each t_i is sandwiched between $-(3d-1) \pm (d-3)$. This completes the proof of Proposition 2.2. $\square$

Example 2.4. Assume $d = 4$, then

$$\frac{1}{2}\, u_2 = \lambda_2 - \lambda_1^2, \quad u_3 = \lambda_3 - \lambda_1\, \lambda_2, \quad 3\, u_4 = \lambda_4 + 2\, \lambda_1\, \lambda_3 - 3\, \lambda_2^2;$$

and

$$\begin{aligned} v_5 &= 3\, \lambda_1\, u_4 - 3\, \lambda_2\, u_3 + \lambda_3\, u_2, \\ v_6 &= 6\, \lambda_2\, u_4 - (2\, \lambda_3 + 6\, \lambda_1\, \lambda_2)\, u_3 + 3\, \lambda_2^2\, u_2. \end{aligned}$$

Hence

$$\begin{aligned} \xi_5 &= V_5 - \lambda^3\, U_2 + 3\, \lambda^2\, U_3 - 3\, \lambda\, U_4, \\ \xi_6 &= V_6 - 3\, \lambda^4\, U_2 + 8\, \lambda^3\, U_3 - 6\, \lambda^2\, U_4. \end{aligned}$$

We have an identity

$$\underbrace{\begin{bmatrix} -1 & 3\, \lambda^{-1} \\ -3\, \lambda & 8 \end{bmatrix}}_{Q} \begin{bmatrix} \xi'_{-6} \\ \xi'_{-5} \end{bmatrix} = \lambda^{-11} \begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix}.$$

Now, $Q = E^{-1}\, D\, F$ for

$$E = \begin{bmatrix} 3\, \lambda & -1 \\ -1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 1 & -3\, \lambda^{-1} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and hence

$$\begin{bmatrix} \xi'_{-5} \\ -3\, \lambda^{-1}\, \xi'_{-5} + \xi'_{-6} \end{bmatrix} = \begin{bmatrix} \lambda^{-11} & 0 \\ 0 & \lambda^{-11} \end{bmatrix} \begin{bmatrix} 3\, \lambda\, \xi_5 - \xi_6 \\ -\xi_5 \end{bmatrix},$$

which gives an isomorphism of $\mathcal{H}$ with $\mathcal{O}_{\mathbf{P}^1}(-11) \oplus \mathcal{O}_{\mathbf{P}^1}(-11)$.

3. Syzygies in the ring of covariants

3.1. Fix an integer q in the range $1 \leqslant q \leqslant e_d - 1$. The following technical result relates the magnitude of α_q to the existence of syzygies in the ring $\mathcal{A}$.

Lemma 3.1. *For an integer $m \geqslant 3$, the following conditions are equivalent:*

(i) $m \geqslant \alpha_q$.

(ii) *Given any covariant Φ of degree-order $(m-2, n)$, and any integer r such that $0 \leqslant r \leqslant \min(2d - 4q - 4, n)$, there exists an identity of the form*

$$(\mathbb{H}_{2q+2}, \Phi)_r = \sum_{i=1}^{q} (\mathbb{H}_{2i}, \Psi_i)_{2(q-i+1)+r+\frac{1}{2}(n_i - n)}, \tag{15}$$

for some covariants Ψ_i of degree-orders $(m-2, n_i)$.

Broadly speaking, condition (ii) means that any expression of the form $(\mathbb{H}_{2q+2}, \square)_\star$ can be rewritten as a sum of terms of the form $\{(\mathbb{H}_{2i}, \square)_\star\}_{1 \leqslant i \leqslant q}$ using algebraic relations in the ring $\mathcal{A}$. The index of transvection of the term $(\mathbb{H}_{2i}, \Psi_i)$ is determined by the requirement that each summand should have order $2d - 4q - 4 + n - 2r$ in $\mathbf{x}$.

Example 3.2. Assume $d = 4$, and let $(q, m) = (1, 3)$. The only choice for Φ (up to a constant) is $\mathbb{F}$, and since $\mathbb{H}_4$ is an invariant, $r = 0$. We have an identity $\mathbb{H}_4 \, \mathbb{F} = 6 \, (\mathbb{H}_2, \mathbb{F})_2$ (see [7, §93]), hence condition (ii) is satisfied. This shows that $\alpha_1 = 3$.

Example 3.3. Assume $d = 7$. The space $\mathcal{A}_{3,9}$ is two dimensional, and it is easy to show (say by specialising $\mathbb{F}$) that $\{(\mathbb{H}_4, \mathbb{F})_2, (\mathbb{H}_2, \mathbb{F})_4\}$ is a basis. Hence there is no identity of the type (15) for $(q, m, r) = (1, 3, 2)$ and $\Phi = \mathbb{F}$, which shows that $\alpha_1 > 3$.

On the other hand, if one takes $(q, m) = (1, 4)$, then such identities always exist. For instance, if $\Phi = \mathbb{H}_6$ and $r = 2$, then

$$(\mathbb{H}_4, \mathbb{H}_6)_2 = \frac{42}{13} \, (\mathbb{H}_2, \mathbb{F}^2)_{10} + \frac{15876}{845} \, (\mathbb{H}_2, \mathbb{H}_2)_8 + \frac{10332}{715} \, (\mathbb{H}_2, \mathbb{H}_4)_6.$$

This can be verified by the use of symbolic calculus as in [7, Ch.V].

PROOF OF LEMMA 3.1. Let $\mathcal{U}$ denote the image of the morphism

$$W_{2q+2} \otimes R_{m-2} \longrightarrow R_m.$$

By definition, it is spanned by all the coefficients of all the transvectants of the form $(\mathbb{H}_{2q+2}, \Phi)_r$. Similarly $(J_{2q})_m$ is spanned by the union of images of the maps

$$W_{2i} \otimes R_{m-2} \longrightarrow R_m, \quad (1 \leqslant i \leqslant q).$$

The inequality $m \geqslant \alpha_q$ holds iff $\mathcal{U}$ is contained in $(J_{2q})_m$, which happens iff an arbitrary $(\mathbb{H}_{2q+2}, \Phi)_r$ can be rewritten as in (15). This proves the lemma. $\square$

3.2. For what it is worth, the lemma gives some thematic support to Conjecture 1.5. Indeed, as m is held constant and q decreases, the range of allowable values of r increases and hence, *prima facie*, condition (ii) becomes more stringent. This makes it plausible that α_q should increase (or at least remain stationary) with decreasing q.

3.3. The next four propositions are the ingredients in Theorem 1.3. In each case we establish a lower bound on some α_q by showing that a certain type of syzygy cannot exist in $\mathcal{A}$ for sufficiently large d.

Proposition 3.4. *If $d \geqslant 12$, then $\alpha_2 > 3$.*

PROOF. Let $(q,m) = (2,3), \Phi = \mathbb{F}$, and $r = 6$ in the notation of Lemma 3.1. To show that condition (ii) fails, it is enough to show that the set

$$\Gamma_1 = (\mathbb{H}_6, \mathbb{F})_6, \quad \Gamma_2 = (\mathbb{H}_4, \mathbb{F})_8, \quad \Gamma_3 = (\mathbb{H}_2, \mathbb{F})_{10},$$

is linearly independent. Specialise to the form

$$F = x_1^d + x_1^{d-2}\, x_2^2 + x_1\, x_2^{d-1} + x_2^d,$$

and calculate the Γ_i. Construct a 3×3 matrix M whose i-th row sequentially consists of the coefficients of

$$x_1^{2d-12}\, x_2^{d-12}, \quad x_1^{2d-13}\, x_2^{d-11}, \quad x_1^{2d-15}\, x_2^{d-9}$$

in Γ_i. For instance, the $(2,1)$-entry is

$$\frac{(d-8)\,(d-9)\,(d-10)\,(d-11)}{8\,(2\,d-9)\,(2\,d-11)\,(2\,d-13)\,(2\,d-15)}.$$

Now $\det(M)$ is a rational function in d, and one easily checks (in MAPLE) that it is nonzero for $d \geqslant 12$. □

One needs to expend a certain quantity of trial and error to discover that $r = 6$ would make the proof work. The analogous argument fails for the set

$$(\mathbb{H}_6, \mathbb{F})_r, \quad (\mathbb{H}_4, \mathbb{F})_{r+2}, \quad (\mathbb{H}_2, \mathbb{F})_{r+4},$$

if $r = 0, 1, 2, 3, 4, 5$. Similar remarks apply to the results below.

Proposition 3.5. *If $d \geqslant 6$, then $\alpha_1 > 3$.*

PROOF. It is enough to show that $(\mathbb{H}_4, \mathbb{F})_2$ is not a constant multiple of $(\mathbb{H}_2, \mathbb{F})_4$ for $d \geqslant 6$. This is done by specialising to the same F as above. □

Proposition 3.6. *If $d \geqslant 16$, then $\alpha_3 > 3$.*

It is enough to show that $(\mathbb{H}_8, \mathbb{F})_{10}$ cannot be written as a linear combination of

$$(\mathbb{H}_6, \mathbb{F})_{12}, \quad (\mathbb{H}_4, \mathbb{F})_{14}, \quad (\mathbb{H}_2, \mathbb{F})_{16}, \tag{16}$$

which can be checked by specialising to $F = x_1^d + x_1^{d-3}\, x_2^3 - x_1\, x_2^{d-1} + 2\, x_2^d$. The details are similar to above. However, this argument works only for $d \geqslant 18$. If $d = 16, 17$, then unfortunately $(\mathbb{H}_8, \mathbb{F})_{10}$ is linearly dependent

on the three covariants in (16), hence one has to look for specific features of those cases.

Assume $d = 16$ or 17, and let $\Phi = \mathbb{F}$ and $(q, r) = (3, 16)$. One can check by specialisation that the covariant $(\mathbb{H}_8, \mathbb{F})_{16}$ does not vanish identically for $d = 16, 17$. It is clear that no relation of the type (15) can exist, since the index of transvection in each summand on the right must be at least 18, which is impossible. This completes the proof. $\square$

Proposition 3.7. *For $d \geqslant 8$, we have $\alpha_1 > 4$.*

PROOF. It is enough to show that there is no constant $\eta_d \in \mathbf{Q}$ such that

$$\underbrace{(\mathbb{H}_4, \mathbb{H}_4)_{2d-8}}_{J} = \eta_d \, \underbrace{(\mathbb{H}_2, \mathbb{H}_2)_{2d-4}}_{K}.$$

Let

$$F_1 = x_1^d + x_2^d, \quad F_2 = x_1^d + x_1^{d-2}\, x_2^2 + x_1\, x_2^{d-1},$$

and consider the determinant $\begin{vmatrix} J_1 & J_2 \\ K_1 & K_2 \end{vmatrix}$, where J_i, K_i denote the specialisations of those invariants to F_i. It is enough to show that this determinant does not vanish for any $d \geqslant 8$. An explicit calculation shows that up to a nonzero factor, it equals

$$f(d) = \underbrace{(d^3 - 8\,d^2 + 19\,d - 14)}_{T_1} + (-1)^d \underbrace{\binom{2d-6}{d-3}}_{T_2}.$$

There is nothing to show for even d, so assume it to be odd. Now $d^4 > T_1$ (because $d^4 - T_1$ has no real roots) and $T_2 > 2^{d-3}$. For $d \geqslant 21$, we have $2^{d-3} > d^4$, and hence $f(d) \neq 0$. Thus it only remains to verify the claim for $d = 9, 11, \ldots, 19$, which is routine. $\square$

In general, let $G^{(q)} = (\mathbb{H}_{2q}, \mathbb{H}_{2q})_{2d-4q}$, which is a degree 4 invariant of d-ics, moreover the $\{G^{(q)}\}$ span the space $\mathcal{A}_{4,0}$. One can deduce a formula for the number $h(d) = \dim \mathcal{A}_{4,0}$ as follows. By Hermite reciprocity (see [14, §157]), it is the same as the number of linearly independent invariants of degree d for binary quartics. If $\mathbb{F}$ denotes the generic quartic, then each such invariant is necessarily of the form $[(\mathbb{F}, \mathbb{F})_4]^a\, [(\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_4]^b$. Hence $h(d)$ is the cardinality of the set

$$\{(a, b) \in \mathbf{N}^2 : 2\,a + 3\,b = d\}.$$

This gives the following formula: write $d = 6\,e + k$ where $0 \leqslant k \leqslant 5$. Then $h(d) = e + \delta_k$, where $\delta_1 = 0$ and $\delta_k = 1$ for $k \neq 1$. For instance, $h(75) = 13$.

Proposition 3.8. *In the saturation sequence of d, at least $h(d)$ of the integers are strictly greater than 4.*

PROOF. Assume that $G^{(q_i)}, (i = 1, 2, \dots, h)$ are linearly independent. Then it is immediate that each $\alpha_{q_i} > 4$. □

The results in this section, a little scattered and unsystematic as they are, should be illustrative of the principle that in so far as the syzygies in $\mathcal{A}$ are intricate and unruly (e.g., see [7, Ch. VII] or [2]), it seems unlikely that one can deduce precise formulae for the α_q.

4. GORDAN'S SYZYGIES

We begin with an explanation of Gordan's cubic syzygies (see [7, §54]). They will be used to prove Theorem 1.4.

4.1. Let f, ϕ, ψ denote binary forms of orders m, n, p respectively; and let a_1, a_2, a_3 be nonnegative integers such that

$$a_2 + a_3 \leqslant m, \quad a_1 + a_3 \leqslant n, \quad a_1 + a_2 \leqslant p.$$

Assume furthermore, that at least one of the following conditions is true:

$$a_1 = 0, \quad \text{or} \quad a_2 + a_3 = m.$$

Then Gordan's syzygy (or series) is the identity

$$\begin{aligned} &\sum_{i=0}^{\infty} \frac{\binom{n-a_1-a_3}{i}\binom{a_2}{i}}{\binom{m+n-2a_3-i+1}{i}} \, ((f,\phi)_{a_3+i}, \psi)_{a_1+a_2-i} \\ = \; &(-1)^{a_1} \sum_{i=0}^{\infty} \frac{\binom{p-a_1-a_2}{i}\binom{a_3}{i}}{\binom{m+p-2a_2-i+1}{i}} \, ((f,\psi)_{a_2+i}, \phi)_{a_1+a_3-i}. \end{aligned}$$

It is usually denoted by $\begin{pmatrix} f & \phi & \psi \\ m & n & p \\ a_1 & a_2 & a_3 \end{pmatrix}$. By convention, $\binom{a}{b} = 0$ if $a < b$, hence either side is a finite sum. The total index of transvection in each term is $a_1 + a_2 + a_3$, which is also called the *weight* of the syzygy. In the following two sections we will specialise to the case $f = \phi = \psi = \mathbb{F}$, and rewrite the syzygies in a more convenient form.

Let $\{a, b\}$ denote the cubic covariant $((\mathbb{F}, \mathbb{F})_a, \mathbb{F})_b$ of order $3d - 2(a + b)$. It vanishes identically unless

$$0 \leqslant a,\ b \leqslant d, \quad a \text{ is even} \quad \text{and} \quad 2a + b \leqslant 2d. \tag{17}$$

An *admissible pair* (a, b) is one which satisfies the conditions in (17). (However, these conditions do not guarantee that $\{a, b\}$ is nonzero; e.g., if $d = 5$, then $\{2, 5\}$ vanishes identically – see [7, §71].)

4.2. **Syzygies of weight at most** d**.** Choose integers w, k in the range

$$0 \leqslant w \leqslant d, \quad 0 \leqslant k < \frac{w}{2},$$

and let $a_1 = 0,\ a_2 = k,\ a_3 = w - k$. Then we have a syzygy

$$\mathfrak{G}_\bullet(k, w): \sum_{m=k}^{w} \theta_{d,k,w}^{(m)} \{m, w - m\} = 0, \tag{18}$$

where

$$\theta_{d,k,w}^{(m)} = \frac{\binom{d-k}{m-k}\binom{w-k}{m-k}}{\binom{2d-k-m+1}{m-k}} - \underbrace{\frac{\binom{d-w+k}{m-w+k}\binom{k}{m-w+k}}{\binom{2d-w+k-m+1}{m-w+k}}}_{(\star)}.$$

The term $(\star)$ is understood to be zero if $m < w - k$. For instance, if $d = 7$, then $\mathfrak{G}_\bullet(1, 6)$ is the syzygy

$$\frac{5}{2}\{2, 4\} + \frac{5}{3}\{4, 2\} - \frac{11}{28}\{6, 0\} = 0.$$

4.3. **Syzygies of weight at least** d**.** Alternately, choose integers w, k in the range

$$d \leqslant w \leqslant \frac{3\,d}{2}, \quad w - d \leqslant k \leqslant \frac{d}{2},$$

and let $a_1 = w - d,\ a_2 = d - k,\ a_3 = k$. Then we have a syzygy

$$\mathfrak{G}^\bullet(k, w): \sum_{m=k}^{2d-w} \vartheta_{d,k,w}^{(m)} \{m, w - m\} = 0, \tag{19}$$

where

$$\vartheta_{d,k,w}^{(m)} = \frac{\binom{2d-w-k}{m-k}\binom{d-k}{m-k}}{\binom{2d-k-m+1}{m-k}} + (-1)^{w+d+1} \underbrace{\frac{\binom{d-w+k}{m-d+k}\binom{k}{m-d+k}}{\binom{d-m+k+1}{m-d+k}}}_{(\star\star)}.$$

The term $(\star\star)$ is understood to be zero if $m < d-k$. For instance, if $d = 11$, then $\mathfrak{G}^\bullet(4,13)$ is the syzygy

$$\{4,9\} + \frac{35}{13}\{6,7\} - \frac{31}{66}\{8,5\} = 0.$$

The syzygies $\mathfrak{G}^\bullet(k,d)$ and $\mathfrak{G}_\bullet(k,d)$ are identical.

4.4. Let us prove Theorem 1.4 for $s = 1$, which claims that α_{e_d-1} is always equal to 3. First, assume d is even, then $\mathbb{H}_d$ is an invariant. It is sufficient to show the existence of a syzygy (15) for $\Phi = \mathbb{F}$ and $r = 0$. This follows from the fact that the coefficient of $\{d,0\}$ in $\mathfrak{G}_\bullet(1,d)$ is

$$\theta^{(d)}_{d,1,d} = \frac{1}{d} - \frac{1}{2} \neq 0.$$

If d is odd, consider the coefficients of $\{d-1,0\}, \{d-1,1\}, \{d-1,2\}$ in the syzygies $\mathfrak{G}_\bullet(1,d-1), \mathfrak{G}_\bullet(1,d)$ and $\mathfrak{G}^\bullet(1,d+1)$ respectively. They are

$$\frac{6}{d(d+1)} - \frac{1}{2}, \quad \frac{6\,(d-1)}{d\,(d+1)} - 1, \quad \frac{6}{d(d+1)} + 1,$$

none of which can be zero. This completes the argument. □

4.5. The following example should illustrate the idea behind the proof of Theorem 1.4. Suppose we want to show that $\alpha_2 = 3$ for $d = 9$. This requires showing (amongst other things) that $\{6,2\}$ can be written as a linear combination of $\{2,6\}$ and $\{4,4\}$. However, any of the Gordan syzygies involving $\{6,2\}$ will also involve the unwanted term $\{8,0\}$. One can use two syzygies simultaneously in order to eliminate the latter. For instance, $\mathfrak{G}_\bullet(1,8)$ and $\mathfrak{G}_\bullet(2,8)$ can be written as

$$\begin{aligned}
\frac{49}{33}\{6,2\} - \frac{13}{30}\{8,0\} &= -\frac{7}{2}\{2,6\} - \frac{70}{13}\{4,4\}, \\
\frac{13}{22}\{6,2\} - \frac{13}{60}\{8,0\} &= -\{2,6\} - \frac{105}{26}\{4,4\}.
\end{aligned}$$

Since the determinant $\begin{vmatrix} 49/33 & -13/30 \\ 13/22 & -13/60 \end{vmatrix}$ is nonzero, $\{6,2\}$ is expressible as a linear combination of $\{2,6\}$ and $\{4,4\}$. The argument in the general case is conceptually the same, but the technical details are somewhat tedious.

4.6. Given an admissible pair (a, b), define its *position* $p(a, b)$ to be the number of admissible pairs (a', b') of the same weight such that $a \leqslant a'$. In any Gordan syzygy involving $\{a, b\}$, it is the $p(a, b)$-th term from the right. For instance, if $d = 13$, then the sequence $(6, 9), (8, 7), (10, 5)$ shows that $p(6, 9) = 3$.

Fix a positive integer s. Our object is to find an integer N_s such that $\alpha_{e_d - s} = 3$ for $d \geqslant N_s$. We will assume that $d \geqslant 4s - 2$; this will prove useful in manipulating the syzygies. (We are making no attempt to find the optimal value of N_s.) First, assume d to be even, say $d = 2n$. Then $\mathbb{H}_{2(n-s+1)}$ has order $4s - 4$, and hence the possible candidates for the left-hand side of (15) are

$$\{2\,(n - s + 1), t\}, \quad \text{for } 0 \leqslant t \leqslant \min(d, 4s - 4) = 4s - 4.$$

Let $w = 2\,(n - s + 1) + t$.

Case I. Assume $0 \leqslant t \leqslant 2s - 2$, then $w \leqslant d$. It is easy to see that the position $p = p(2n - 2s + 2, t)$ equals $[\frac{t}{2}] + 1$. Construct a $p \times p$ matrix M_t whose (k, m)-th entry is $\theta^{(2m)}_{d,k,w}$, for

$$1 \leqslant k \leqslant p, \qquad n - s + 1 \leqslant m \leqslant n - s + p.$$

Case II. Assume $2s - 1 \leqslant t \leqslant 4s - 4$, then $d + 1 \leqslant w \leqslant \frac{3d}{2}$ and $p = 2s - 1 - \lceil \frac{t}{2} \rceil$. Construct M_t by letting its (k, m)-th element to be $\vartheta^{(2m)}_{d,k,w}$, for

$$w - d \leqslant k \leqslant w - d + p - 1, \qquad n - s + 1 \leqslant m \leqslant n - s + p.$$

Now let d be odd, say $d = 2n + 1$. Then $\mathbb{H}_{2(n-s+1)}$ has order $4s - 2$, and one can construct matrices M'_t as above for $0 \leqslant t \leqslant 4s - 2$. It is clear that

$$\Delta_t(d) = \det M_t, \quad \Delta'_t(d) = \det M'_t,$$

are rational functions of d. I have calculated them explicitly for $s \leqslant 8$, and in each case determined the threshold N_s such that they are all nonzero for $d \geqslant N_s$. The computations were programmed in MAPLE. For instance[5], if $s = 3$, then

$$\Delta_6(d) = \frac{3780\,(d-4)\,(d-5)\,(d-6)\,(d+7)\,(d^2+3\,d+10)}{(d-1)^2\,(d-2)\,(d+2)\,(d+1)^2\,d^2\,(d+3)},$$

which is nonzero for $d > 6$.

[5]It seems to be a general feature that the numerators and denominators of Δ_t, Δ'_t almost entirely consist of linear factors. Why this should be so is not obvious to me.

As in the example above, this shows the existence of a syzygy for each $\{2(n-s+1), t\}$ as required by (15). $\square$

The argument would break down if any of the determinants were to vanish identically; but fortunately this does not happen, at least for $s \leqslant 8$. The theorem could be mechanically extended to a few more values of s, but this is unlikely to be of much interest in itself. This line of argument suggests the following conjecture.

Conjecture 4.1. *For any positive integer s, there exists an integer N_s such that (at least) the last s integers in the saturation sequence are equal to 3 for all $d \geqslant N_s$.*

This would follow immediately if it could be shown that Δ_t, Δ_t' never vanish identically. Furthermore, the data suggest that $N_s = 4\,s - 2$ is in fact the best possible value for $s \geqslant 3$.

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